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PSEUDOCOMPLEMENTED RESIDUAL LATTICE

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SECTIONALLY PSEUDOCOMPLEMENTED RESIDUAL LATTICE

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Abstract: At first, we recall the basic concept. By a residual lattice is meant an algebra $L = (L; \vee, \wedge, *, \circ, 0, 1)$ such that

(i) $L = (L; \vee, \wedge, 0, 1)$ is a bounded lattice,
(ii) $L = (L; *, 1)$ is a commutative monoid,
(iii) it satisfies the so-called adjoinness property:
$$y \circ z = y \iff y \leq z \leq x \circ y$$

Let us note [7] that $x \vee y$ is the greatest element of the set $\{x \vee y\}$. Moreover, if we consider $x \ast y = x \wedge y$, then $x \circ y$ is the relative pseudo-complement of $x$ with respect to $y$, i.e., for $\ast = \wedge$ residuated lattices are just relatively pseudo-complemented lattices. The identities characterizing sectionally pseudo-complemented lattices are presented in [3] i.e. the class of these lattices is a variety in the signature $\{\vee, \wedge, \ast, \circ, 0\}$. We are going to apply a similar approach for the adjointness property.

Key words: Residuated lattice, non Distributive, Residuated Abelian, commutative monoid.

1. Introduction

Residuated lattices were introduced by Ward and Dilworth [5] and studied by several authors. Two monographs contain a compendium on residuated lattices. They are that by Blyth and Janowitz [1] (where it is renamed as a residuated Abelian semi-group with a unit) and the book by R. Belohavek [7]. In this short note we will compare a certain modification of a residuated lattice with already introduced [2], [3]. At first, we recall the basic concept:

Definition 1. A lattice $L = (L; \vee, \wedge, \ast)$ with the greatest element 1 is sectionally pseudo-complemented if each interval $[y, 1]$ is a pseudo-complemented lattice.

From now on, denote by $x \vee y$ the pseudo-complement of $x \vee y$ in the interval $[y, 1]$. Naturally, $x \vee y \in [y, 1]$ thus $L = (L; \vee, \wedge, \ast, \circ, 0, 1)$ is sectionally pseudo-complemented if and only if “o” is an (everywhere defined) operation on $L$.

Definition 2. An algebra $L = (L; \vee, \wedge, \ast, \circ, 0, 1)$ is called a sectionally residuated lattice if

(i) $L = (L; \vee, \wedge, 0, 1)$ is a lattice with the greatest element 1;
(ii) $L = (L; *, 1)$ is a commutative monoid;
(iii) it satisfies the sectional adjointness property: $(x \vee y) \circ z = y \iff y \leq z \leq x \circ y$

Lemma 1.1 Let $L = (L; \vee, \wedge, \ast, \circ, 0, 1)$ be a sectionally residuated lattice. Then $x \ast y$ is the greatest element of the set $\{x \ast y\}$. This immediately yields the following facts:

$$x \ast y \ast (x \circ y) = y,$$  \hspace{1cm} (1)

$$x \vee (x \ast y) = y,$$ \hspace{1cm} (2)

$$y \leq x \circ y,$$ \hspace{1cm} (3)

Lemma 1.2 Let $L = (L; \vee, \wedge, \ast, \circ, 0, 1)$ be a sectionally residuated lattice. Then $x \leq y$, if and only if $x \circ y = 1$

Proof: Suppose $x \leq y$, then $x \vee y = y$, and by Lemma 1.1, $x \circ y$ is the greatest element of the set $\{x \ast y \ast z = y\}$ By Definition 2, $y \ast 1 = 1$ thus $x \circ y = 1$. Conversely, suppose $x \circ y = 1$. Then, by [1], we have $y = (x \vee y) \ast (x \circ y) = (x \vee y) \ast 1 = x \vee y$.

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Lemma 1.3 In a sectionally residuated lattice, the following identities are satisfied:
and 1 \circ x = x 

Proof: The first three identities follow directly by Lemma 1.2. Further, by Lemma 1.1, 1 \circ x is the greatest element of the set \{z; 1 \circ z = x\} = \{x\} thus 1 \circ x = x

Lemma 1.4 In a sectionally residuated lattice, \(ab = a\) if and only if \(a = b\)

Proof: Putting \(a = x = a\) and \(z = b\) in the sectional adjointness property, the assumption \(ab = a\) yields \((a \lor a) \circ b\) iff \(a \leq b \leq a \circ a = 1\) thus \(a \leq b\)

Conversely, \(a \leq b\) implies by Lemma 3 \(a \leq b \leq 1 = a \circ a\) and, by sectional adjointness, \(ab = (a \lor a) \circ b = a\)

Applying Lemma 1.2 and Lemma 1.4, we get

Corollary 1.5 In a sectionally residuated lattice,
(a) \(x \star y = x\) if and only if \(x \star y = 1\);
(b) \(x \star x = x\)

Lemma 1.6 In a sectionally residuated lattice, \(x \land y \leq x \star y\).

Proof: By [3] we have \(x \land y \leq x \circ (x \land y)\).
Applying sectional adjointness, we infer \(x \star (x \land y) = (x \lor (x \land y)) \star (x \land y)\) and, analogously, \(y \star (x \land y) = x \land y\). Hence, by Corollary 1.5 (b),
\[x \star y \star (x \land y) = x \star (x \land y) \star y \star (x \land y) = (x \land y) \star (x \land y) = x \land x\]
and by Lemma 1.4, \(x \land y \leq x \star y\).

Theorem 1.7 Let \(L = (L; \lor, \land, \star, \circ, 1)\) be a sectionally residuated lattice. Then it is a sectionally pseudo-complemented lattice.

Proof: Replacing \(y\) by \(x \land y\) in the sectional adjointness property, we obtain \(x \star z = x \land y\) iff \(x \land y \leq z \leq x \circ (x \land y)\).
However, \(x \circ (x \land y)\) is the greatest element of the set \(\{t; (x \lor (x \land y)) \star t = x \land y\} = \{t; x \star t = x \land y\}\). By Lemma 1.4, \(x \land t \leq x \star t = x \land y\), thus the greatest \(t\) of this property satisfies \(t \geq y\).
Thus \(y \leq x \circ (x \land y)\), i.e., \(x \land y \leq y \leq x \circ (x \land y)\) and by the sectional adjointness, \(x \star y = (x \land (x \lor y)) \star y = x \land y\).
Hence, \(x \circ y\) is the pseudo-complement of \(x \lor y\) in the interval \([y, l]\)

2. Conclusion
It is well known that every relatively pseudo-complemented lattice is distributive.
An extension of relative pseudo-complementation for the non-distributive case was already involved in [3], [4]:

References