SECTIONALLY
PSEUDOCOMPLEMENTED RESIDUAL LATTICE

Rahman, Md. Zaidur
Daffodil International University

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Sectionally Pseudocomplemented Residual Lattice

Md. Zaidur Rahman 1, Md. Abul Kalam Azad 1 and Md. Nazmul Hasan. 2
Dept. of Mathematics 1
Khulna University of Engineering and Technology
Dept. of Mathematics 2
Moheshpur Govt. College, Moheshpur, Jhenaidah.
E-mail mzrahman1968@gmail.com, azadmath.azad8@gmail.com

Abstract: At first, we recall the basic concept. By a residuated lattice is meant an algebra $L = (L; \lor, \land, *, 0, 1)$ such that
(i) $L = (L; \lor, \land, 0, 1)$ is a bounded lattice,
(ii) $L = (L; *, 1)$ is a commutative monoid,
(iii) it satisfies the so-called adjoinness property:
$$y\lor y \lor x = x \land y \land x$$
if and only if $y \leq z \leq x \land y$

Let us note [7] that $x \lor y$ is the greatest element of the set $(x \lor y) \star z = y$.

Moreover, if we consider $x \star y = x \land y$, then $x \land y$ is the relative pseudo-complement of $x$ with respect to $y$, i.e., for $\ast = \land$ residuated lattices are just relatively pseudo-complemented lattices. The identities characterizing sectionally pseudo-complemented lattices are presented in [3] i.e. the class of these lattices is a variety in the signature $\{\lor, \land, \ast\}$. We are going to apply a similar approach for the adjoinness property.

Key words: Residuated lattice, non Distributive, Residuated Abelian, commutative monoid.

1. Introduction

Residuated lattices were introduced by Ward and Dilworth [5] and studied by several authors. Two monographs contain a compendium on residuated lattices. They are that by Blyth and Janowitz [1] (where it is renamed as a residuated Abelian semi-group with a unit) and the book by R. Belohavék [7].

In this short note we will compare a certain modification of a residuated lattice with already introduced [2], [3]. At first, we recall the basic concept:

Definition 1. A lattice $L = (L; \lor, \land, \ast)$ with the greatest element 1 is sectionally pseudo-complemented if each interval $[y, 1]$ is a pseudo-complemented lattice.

From now on, denote by $x \lor y$ the pseudo-complement of $x \lor y$ in the interval $[y, 1]$.

Naturally, $x \lor y \in [y, 1]$ thus $L = (L; \lor, \land, \ast)$ is sectionally pseudo-complemented if and only if "\lor" is an (everywhere defined) operation on L.

Definition 2. An algebra $L = (L; \lor, \land, *, 1)$ is called a sectionally residuated lattice if
(i) $L = (L; \lor, \land, 0, 1)$ is a lattice with the greatest element 1;
(ii) $L = (L, *)$ is a commutative monoid;
(iii) it satisfies the sectional adjoinness property: $(x \lor y) \ast z = y$ if and only if $y \leq z \leq x \land y$

Lemma 1.1 Let $L = (L; \lor, \land, *, 1)$ be a sectionally residuated lattice. Then $x \ast y$ is the greatest element of the set $\{z; (x \lor y) \ast z = y\}$

This immediately yields the following facts:

$$x \land y \ast (x \land y) = y,$$
$$x \lor y \ast y = y,$$
$$y \leq x \land y,$$

Lemma 1.2 Let $L = (L; \lor, \land, *, 1)$ be a sectionally residuated lattice. Then $x \leq y$, if and only if $x \ast y = 1$

Proof: Suppose $x \leq y$, then $x \lor y = y$, and by Lemma 1.1, $x \ast y$ is the greatest element of the set $\{z; y \ast z = y\}$ By Definition 2, $y \ast 1 = 1$ thus $x \ast y = 1$. Conversely,

Suppose $x \ast y = 1$. Then, by [1], we have $y = (x \lor y) \ast (x \land y) = (x \lor y) \ast 1 = x \lor y$
whence \( x \leq y \)

**Lemma 1.3** In a sectionally residuated lattice, the following identities are satisfied: and \( 1 \circ x = x \)

**Proof:** The first three identities follow directly by Lemma 1.2. Further, by Lemma 1.1, \( 1 \circ x \) is the greatest element of the set \( \{x;1 \ast z = x\} = \{x\} \) thus \( 1 \circ x = x \)

**Lemma 1.4** In a sectionally residuated lattice, \( a \ast b = a \) if and only if \( a = b \)

**Proof:** Putting \( x = y = a \) and \( z = b \) in the sectional adjointness property, the assumption \( a \ast b = a \) yields \( (a \lor a) \ast b \) iff \( a \leq b \leq a \circ a = 1 \) thus \( a \leq b \)

Conversely, \( a \leq b \) implies by Lemma 3 \( a \leq b \leq 1 = a \circ a \) and, by sectional adjointness, \( a \ast b = (a \lor a) \ast b = a \)

Applying Lemma 1.2 and Lemma 1.4, we get

**Corollary 1.5** In a sectionally residuated lattice,

(a) \( x \ast y = x \) if and only if \( x \circ y = 1 \)

(b) \( x \ast x = x \)

**Lemma 1.6** In a sectionally residuated lattice, \( x \land y \leq x \ast y \)

**Proof:** By [3] we have \( x \land y \leq x \circ (x \land y) \). Applying sectional adjointness, we infer \( x \ast (x \land y) = (x \lor (x \land y)) \ast (x \land y) \) and, analogously, \( y \ast (x \land y) = x \land y \). Hence, by Corollary 1.5 (b),

\[
\begin{align*}
   x \ast y \ast (x \land y) &= x \ast (x \land y) \ast y \ast (x \land y) \\
   &= (x \land y) \ast (x \land y) = x \land x \\
\end{align*}
\]

and by Lemma 1.4, \( x \land y \leq x \ast y \).

**Theorem 1.7** Let \( L = (L; \lor, \land, \ast, \circ, 1) \) be a sectionally residuated lattice, then it is a sectionally pseudo-complemented lattice.

**Proof:** Replacing \( y \) by \( x \land y \) in the sectional adjointness property, we obtain \( x \ast z = x \land y \iff x \land y \leq z \leq x \circ (x \land y) \).

However, \( x \circ (x \land y) \) is the greatest element of the set \( \{t;(x \lor (x \land y)) \ast t = x \land y\} = \{t;x \ast t = x \land y\} \).

By Lemma 1.4, \( x \land t \leq x \ast t = x \land y \), thus the greatest \( t \) of this property satisfies \( t \geq y \).

Thus \( y \leq x \circ (x \land y) \), i.e., \( x \land y \leq y \leq x \circ (x \land y) \) and by the sectional adjointness, \( x \ast y = (x \land (x \land y)) \ast y = x \land y \).

Hence, \( x \circ y \) is the pseudo-complement of \( x \lor y \) in the interval \([y,l]\)

**2. Conclusion**

It is well known that every relatively pseudo-complemented lattice is distributive.

An extension of relative pseudo-complementation for the non-distributive case was already involved in [3], [4]:

**References**


