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SECTIONALLY PSEUDOCOMPLEMENTED RESIDUAL LATTICE

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Abstract : At first, we recall the basic concept, By a residual lattice is meant an algebra $L = (L; \lor, \land, *, o, 1, 0)$ such that

(i) $L = (L; \lor, \land, 0, 1)$ is a bounded lattice,
(ii) $L = (L; *, 1)$ is a commutative monoid,
(iii) it satisfies the so-called adjoinness property:

$$y \land y \lor x \Rightarrow y \land z \land x \Rightarrow$$

Let us note [7] that $x \lor y$ is the greatest element of the set $(x \lor y) \land z = y$

Moreover, if we consider $x \land y = x \lor y$, then $x \land y$ is the relative pseudo-complement of $x$ with respect to $y$, i.e., for $* = \land$ residuated lattices are just relatively pseudo-complemented lattices. The identities characterizing sectionally pseudo-complemented lattices are presented in [3] i.e. the class of these lattices is a variety in the signature $\{ \lor, \land, 0, 1 \}$. We are going to apply a similar approach for the adjoinness property:

Key words: Residuated lattice, non Distributive, Residuated Abelian, commutative monoid:

1. Introduction
Residuated lattices were introduced by Ward and Dilworth [5] and studied by several authors. Two monographs contain a compendium on residuated lattices. They are that by Blyth and Janowitz [1] (where it is renamed as a residuated Abelian semi-group with a unit) and the book by R. Belohavek [7]. In this short note we will compare a certain modification of a residuated lattice with already introduced [2], [3]. At first, we recall the basic concept:

Definition 1. A lattice $L = (L; \lor, \land, 0, 1)$ with the greatest element $1$ is sectionally pseudo-complemented if each interval $[y, 1]$ is a pseudo-complemented lattice.

From now on, denote by $x \lor y$ the pseudo-complement of $x \lor y$ in the interval $[y, 1]$.

Naturally, $x \lor y \in [y, 1]$ thus $L = (L; \lor, \land, 0, 1)$ is sectionally pseudo-complemented if and only if "o" is an (everywhere defined) operation on $L$.

Definition 2. An algebra $L = (L; \lor, \land, *, o, 1)$ is called a sectionally residuated lattice if

(i) $L = (L; \lor, \land, 0, 1)$ is a lattice with the greatest element $1$;
(ii) $L = (L; *, 1)$ is a commutative monoid ;
(iii) it satisfies the sectional adjoinness property: $(x \lor y) \land z = y$ if and only if $y \leq z \leq x \land y$

Lemma 1.1 Let $L = (L; \lor, \land, *, o, 1)$ be a sectionally residuated lattice. Then $x \land y$ is the greatest element of the set $(x \land y) \lor z = y$

This immediately yields the following facts:

$$\begin{align*}
(x \lor y) \lor (x \land y) &= y, \\
(x \lor y) \lor y &= y, \\
y \leq x \land y,
\end{align*}$$

Lemma 1.2 Let $L = (L; \lor, \land, *, o, 1)$ be a sectionally residuated lattice. Then $x \lor y$, if and only if $x \land y = 1$

Proof: Suppose $x \leq y$, Then $x \lor y = y$, and by Lemma 1.1, $x \land y$ is the greatest element of the set $(x \lor y) \land z = y$ By Definition 2, $x \land y = 1$ thus $x \land y = 1$. Conversely,

Suppose $x \land y = 1$. Then, by [1], we have $y = (x \lor y) \lor (x \land y) = (x \lor y) \lor 1 = x \lor y$
whence \( x \leq y \)

**Lemma 1.3** In a sectionally residuated lattice, the following identities are satisfied:

\[ 1 \circ x = x \]

**Proof:** The first three identities follow directly by Lemma 1.2. Further, by Lemma 1.1,

\[ 1 \circ x \]

is the greatest element of the set \( \{ z ; 1 \circ z = x \} \) thus \( 1 \circ x = x \)

**Lemma 1.4** In a sectionally residuated lattice, \( a \ast b = a \) if and only if \( a = b \)

**Proof:** Putting \( x = y = a \) and \( z = b \) in the sectional adjointness property, the assumption \( a \ast b = a \) yields \( (a \vee a) \ast b \) iff \( a \leq b \leq a \circ a = 1 \) thus \( a \leq b \)

Conversely, \( a \leq b \) implies by Lemma 3 \( a \leq b \leq 1 = a \circ a \) and, by sectional adjointness, \( a \ast b = (a \vee a) \ast b = a \)

Applying Lemma 1.2 and Lemma 1.4, we get

**Corollary 1.5** In a sectionally residuated lattice,

(a) \( x \ast y = x \) if and only if \( x \circ y = 1 \);

(b) \( x \ast x = x \)

**Lemma 1.6** In a sectionally residuated lattice, \( x \land y \leq x \ast y \).

**Proof:** By [3] we have \( x \land y \leq x \circ (x \land y) \). Applying sectional adjointness, we infer \( x \ast (x \land y) = (x \lor (x \land y)) \ast (x \land y) \) and, analogously, \( y \ast (x \land y) = x \land y \). Hence, by Corollary 1.5 (b),

\[ x \ast y \ast (x \land y) = x \ast (x \land y) \ast y \ast (x \land y) = (x \land y) \ast (x \land y) = x \land x \]

and by Lemma 1.4, \( x \land y \leq x \ast y \).

**Theorem 1.7** Let \( L = (L; \lor, \land, \ast, \circ, \leq) \) be a sectionally residuated lattice. Then it is a sectionally pseudo-complemented lattice.

**Proof:** Replacing \( y \) by \( x \land y \) in the sectional adjointness property, we obtain \( x \ast z = x \land y \iff x \land y \leq z \leq x \circ (x \land y) \).

However, \( x \circ (x \land y) \) is the greatest element of the set \( \{ t ; (x \lor (x \land y)) \ast t = x \land y \} = \{ t ; x \ast t = x \land y \} \).

By Lemma 1.4, \( x \land t \leq x \ast t = x \land y \), thus the greatest \( t \) of this property satisfies \( t \geq y \).

Thus \( y \leq x \circ (x \land y) \), i.e., \( x \land y \leq y \leq x \circ (x \land y) \) and by the sectional adjointness, \( x \ast y = (x \land (x \lor y)) \ast y = x \land y \).

Hence, \( x \circ y \) is the pseudo-complement of \( x \lor y \) in the interval \([y, l]\).

2. Conclusion

It is well known that every relatively pseudo-complemented lattice is distributive. An extension of relative pseudo-complementation for the non-distributive case was already involved in [3], [4]:

**References**


